2025

(Nov–Dec)

FYUP : 5th Semester Examination

MAJOR

MATHEMATICS

(Calculus—III)

MTH–300

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **four** questions, selecting **one** from each Unit

UNIT—I

1. (a) Show that the set

$$\left\{1,\ -1,\ 1\frac{1}{2},\ -1\frac{1}{2},\ 1\frac{1}{3},\ -1\frac{1}{3},\ \cdots\right\}$$

is closed in $\mathbb{R}$. 3

(b) State and prove Bolzano-Weierstrass theorem in $\mathbb{R}$. 1+5=6 *(c)* State whether the following statements are True or False with justifications : 3×3=9

(i) Every finite subset of $\mathbb{R}$ is closed.

(ii) Every infinite subset of $\mathbb{Q}$ has a limit point.

(iii) The intersection of any number of open sets is again an open set.

2. *(a)* State and prove Heine-Borel theorem. 1+5=6

(b) Define a compact set. Show that an infinite subset S of $\mathbb{R}$ is compact if and only if every infinite subset thereof has a limit point in the set. 1+5=6

(c) Give examples (with justifications) of the following : 3×2=6

(i) A subset of $\mathbb{R}$ having a limit point but is not bounded in $\mathbb{R}$.

(ii) A bounded subset of $\mathbb{R}$ which is not compact.

UNIT—II

3. *(a)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables defined as follows :

$$f(x, y) = \begin{cases} \dfrac{2x^2 + y^4}{x^2 y}, & x \neq 0,\ y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Test the continuity of f at the origin. 4

(b) State whether the following statements are True or False with justifications : 5+2+3=10

(i) If f is a real-valued continuous function on the closed interval $[a, b]$, then f is bounded.

(ii) Every bounded real-valued function on $[a, b]$ is continuous.

(iii) The real-valued function $f(x) = x^2 + 3x$, for all $x \in [-1, 1]$, is uniformly continuous.

(c) Prove that if a real-valued function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$. 5

4. *(a)* If $f : \mathbb{R} \to \mathbb{R}$ is continuous on a compact set $S \subseteq \mathbb{R}$, then show that $f(S)$ is compact. 5

(b) Let X and Y be subsets of $\mathbb{R}^n$. Show that a function $f : X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is open for every open set $V \subseteq Y$. 6

(c) Show that the real-valued function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$. 6

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $f(x) = 1$, $\forall x \in \mathbb{Q}$. Show that $f(x) = 1$, $\forall x \in \mathbb{R}$. 2

UNIT—III

5. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & \text{if } x - y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Show that at (0, 0), the function f possesses directional derivatives in every direction. Also show that the function is not continuous at (0, 0). 6

(b) Let $f : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$ and $(a, b) \in D$ satisfying—

(i) f_x exists at (a, b);

(ii) f_y is continuous at (a, b).

Then prove that f is differentiable at (a, b). 6

(c) Show that the mean value theorem

$$f(y) - f(x) = f'(z)(y - x)$$

does not hold for the function $f : \mathbb{R} \to \mathbb{R}^2$, given by $f(t) = (\cos t, \sin t)$ for certain values of x and y. 7

6. (a) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$$

5

(b) Prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives at a point, then it is differentiable at that point. 7

(c) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (0, 0) but the partial derivatives are not continuous at this point, where

$$f(x, y) = x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) \text{ if } xy \neq 0$$

$$f(x, 0) = x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0$$

$$f(0, y) = y^2 \sin\left(\frac{1}{y}\right) \text{ if } y \neq 0$$

$$f(0, 0) = 0$$

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UNIT—IV

7. (a) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy\dfrac{(x^2 - y^2)}{(x^2 + y^2)}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

does not satisfy conditions of Schwarz's theorem. 6

(b) Find all the maxima and minima of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$$ 7

(c) Expand, using Taylor's formula, the function $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y) = e^x \cos y$, about the point $\left(1, \frac{\pi}{4}\right)$. 6

8. (a) State and prove Schwarz's theorem for equality of mixed partial derivatives. 2+5=7

(b) If $f(x, y) = |x^2 - y^2| \forall x, y \in \mathbb{R}$, determine whether

$$f_{xy}(0, 0) = f_{yx}(0, 0)$$ 6

(c) Find the maximum value of

$$\left|\sum_{k=1}^{2} a_k x_k\right|, \text{ if } \sum_{k=1}^{2} x_k^2 = 1$$

by using Lagrange's multipliers method, where a_k's are positive constants for $k = 1, 2$. 6

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